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Lipschitz–Killing Curvatures of Angular Partially Ordered Sets

LOTHAR BUDACH

Akademie der Wissenschaften der DDR, Forschungsbereich Mathematik und Informatik, Rudower Chaussee 5, 1199 Berlin, German Democratic Republic

The present paper is motivated by recent work of J. Cheeger, W. Müller, and R. Schrader on the Lipschitz–Killing curvatures of piecewise flat spaces. It is proved that all basic properties of Lipschitz–Killing curvatures can be derived by combinatorial methods using only the Gram–Sommerville and McMullen equations. Angular partially ordered sets yield the appropriate framework for proving the main results by means of the Rota calculus in the incidence algebra of cell complexes. © 1989 Academic Press, Inc.

INTRODUCTION

The present paper is motivated by recent work of J. Cheeger, W. Müller, and R. Schrader [CMS] on the Lipschitz–Killing curvatures of piecewise flat spaces. In order to find analogues to the classical Lipschitz–Killing curvatures of Riemannian manifolds relative to piecewise flat spaces, J. Cheeger [Ch] and P. Wintgen [Wi] generalized definitions of L. A. Brin [Br], T. Regge [Re], and T. Banchoff [Ba] (see also [St] and [Po]). In [CMS] analogues of the basic results for curvatures in differential geometry are proved for curvatures of piecewise flat spaces.

Once the definition of Lipschitz–Killing curvatures for such quasicombinatorial objects as arbitrary piecewise flat spaces is realized, interest will naturally center upon the question whether one can prove also the main results by combinatorial methods only.

Many a problem for Lipschitz–Killing curvatures is mainly of combinatorial nature. But in [CMS] very often this combinatorial nature is hidden behind an analytic argumentation. The same argumentation is used in the review [La] of J. Lafontaine in Séminaire Bourbaki. So it appeared worth indicating the possibility of a combinatorial representation of basic results for Lipschitz–Killing curvatures not using analytic arguments. This is the aim of the present note. Elementary combinatorial proofs of all basic facts for Lipschitz–Killing curvatures are given in the axiomatic frame of so-called angular partially ordered sets introduced in Section 4.

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The paper is organized into six sections as follows: We begin in Section 1 with a brief introduction of the basic geometric facts in order to enable the reader in all parts of our paper to see the geometric motivation behind the combinatorial techniques.

Section 2 contains the combinatorial tools: the incidence algebra of a locally finite partially ordered set and the combinatorial interpretation of the basic geometric identities of D. M. Y. Sommerville and P. McMullen.

In Section 3 the Chern–Gauss–Bonnet densities are introduced. The main result of this section is the elementary proof of Theorem 3.11 which J. Cheeger proved by heat equation methods. In our proof all properties follow essentially from the relations of Gram–Sommerville and McMullen.

In Section 4 Lipschitz–Killing curvatures of arbitrary angular partially ordered sets are introduced and the additivity of these curvatures is verified (Theorem 4.8).

The behavior of Lipschitz–Killing curvatures under metric products is studied in Section 5 (Proposition 5.2 and Theorem 5.4).

Section 6 proves that Lipschitz–Killing curvatures are combinatorial invariants, i.e., that they remain unchanged by subdivisions. This allows us to introduce in Section 7 Lipschitz–Killing curvatures as invariants of arbitrary polyhedrons.

For applications of Lipschitz–Killing curvatures in computation theory which have been the original motivation for the present note, we refer the reader to the author's paper [Bu].

1. Polytopes, Complexes, and Angles

In this section the geometric background of all considerations of our paper will be introduced. For details we refer the reader to [MS], [RS].

Convex Polytopes

1.1. The convex hull of a finite set of points of the Euclidean d-space $\mathbf{R}^d$ is called a *convex polytope*. A convex polytope P which spans a subspace $\langle P \rangle$ of dimension n is called an *n-polytope* or an *n-cell*. Define P°, $\dot{P}$ to be the interior and frontier of P in $\langle P \rangle$. An n-cell which is the convex hull of $n+1$ points is called an *n-simplex*.

1.2. Let $H := \{x \in \mathbf{R}^d \mid \langle x, a \rangle = \lambda\}$ be a hyperplane in $\mathbf{R}^d$ ($a \neq 0$ is a vector of R^d, $a = (a_1, ..., a_n)$, $x = (x_1, ..., x_n)$, and $\langle x, a \rangle := \sum_{i=1}^d x_i a_i$ is the inner product). The two closed half-spaces determined by H are

$$H^+ := \{x \in \mathbf{R}^d \mid \langle x, a \rangle \geqslant \lambda\}$$
$$H^- := \{x \in \mathbf{R}^d \mid \langle x, a \rangle \leqslant \lambda\}.$$

Let P be a convex polytope in $\mathbf{R}^d$. Then a hyperplane H of $\mathbf{R}^d$ is called a *supporting hyperplane* of P if $H \cap P \neq \emptyset$, and P is contained in one of the closed half-spaces determined by H. If $x \in H \cap P$, then we say H supports P at x. If $P \subseteq H^+$ then $-a$ is called the *outward normal vector* of P at x; if $P \subseteq H^-$, then a is an outward normal vector of P at x.

1.3. THEOREM. *Let P be a convex polytope and a be a nonzero vector. Then a is an outward normal vector of P. A convex polytope is the intersection of a finite number of closed half-spaces belonging to supporting hyperplanes.*

For a proof see [RS, Theorem 12 of 1.3 and Theorem 3 of 2.1].

Faces

1.4. Let H be a supporting hyperplane of a convex polytope P. Then $H \cap P$ is called a *face* of P. We include $\emptyset$ and P among the faces. A j-cell which is a face is called a j-face. If P is an n-cell then the $(n-1)$-faces of P are called *facets* of P, 1-faces are called *edges*. The 0-faces are called *vertices* of P and the set of all vertices of P is denoted by vert P. $\emptyset$ is called an 1-face.

1.5. THEOREM. *A convex polytope has only a finite number of faces and each face is a convex polytope. An n-cell P has faces of each dimension $j = -1, 0, 1, 2, \ldots, n$. The set $\mathscr{F}(P)$ of faces of P, partially ordered by inclusion, is a finite lattice, called the* face lattice *of P.*

See [MS, 2.1 and 2.2] for proofs.

Notation. If F is a face of P then we write $F \leqslant P$, if F is a proper face of P, i.e., if also $F \neq P$ then we write $F < P$. Remark that

$$P = \bigsqcup_{F \leqslant P} F^\circ,$$

$$\dot{P} = \bigsqcup_{F < P} F^\circ,$$

where $\bigsqcup$ denotes the disjoint union.

Angles

1.6. Let P be an n-cell in $\mathbf{R}^d$ and F an n-face of P. Let $\mathbf{C}^\perp(F, P)$ be the normal cone of F in P consisting of all rays through the centroid z of F (or any other point of F) which are orthogonal to F and point into P. Let $N(F, P)$ be the vector space of dimension $n-m$ of all vectors through z which are orthogonal to F and point into $\langle P \rangle$. Let S^{n-m-1} be the unit sphere consisting of all vectors of $N(F, P)$ of length 1. The interior angle

$\alpha(F, P)$ of P at F (or at F in P) is the ratio of the $(n-m-1)$-content of $S^{n-m-1} \cap \mathbf{C}^{\perp}(F, P)$ to the $(n-m-1)$-content of S^{n-m-1}:

$$\alpha(F, P) := \frac{\operatorname{vol}(S^{n-m-1} \cap \mathbf{C}^{\perp}(F, P))}{O_{n-m-1}},$$

where $O_n := \operatorname{vol} S^n$.

Let $\mathbf{C}^{\perp}(F, P)^*$ be the dual normal cone of F in P consisting of all rays through z making an angle $>\pi/2$ with all rays of $\mathbf{C}^{\perp}(F, P)$. The exterior angle $\beta(F, P)$ of P at F (or at F in P) is the normalized content of $S^{n-m-1} \cap \mathbf{C}^{\perp}(F, P)^*$:

$$\beta(F, P) := \frac{\operatorname{vol}(S^{n-m-1} \cap \mathbf{C}^{\perp}(F, P)^*)}{O_{n-m-1}}.$$

$\alpha(F, P)$ and $\beta(F, P)$ are independent of the choice of z. We assume

$$\alpha(F, F) = \beta(F, F) = 1.$$

$\alpha(F, P)$ can be defined also in an alternative way. Let $\mathbf{C}(F, P)$ be the cone with the vertex at any point z of F (f.i. the centroid of F) spanned by P. Let S^{n-1} be the unit sphere consisting of all vectors of $\langle P \rangle$ of length 1. $\alpha(F, P)$ is the ratio of the $(n-1)$-content of $S^{n-1} \cap \mathbf{C}(F, P)$ to the $(n-1)$-content of S^{n-1}:

$$\alpha(F, P) := \frac{\operatorname{vol}(S^{n-1} \cap \mathbf{C}(F, P))}{O_{n-1}}.$$

It is easy to see that these two definitions of the interior angle $\alpha(F, P)$ coincide (see f.i. [Ba, Lemma 2]).

1.7. Lemma. $\sum_{X \in \operatorname{vert} P} \beta(X, P) = 1$ *for every convex polytope P.*

Proof. If $L(X, P) := C^*(X, P) \cap S^{n-1}$ for $X \in \operatorname{vert} P$ then $L(X, P)$ consists of all outward normal unit vectors of P at X. Let a be a unit vector of S^{n-1}. Then by 2.3, a is an outward normal vector of P. Let H be the supporting hyperplane of P with the outward normal vector a. Then $H \cap P$ is a face of P. Let X be a vertex of $H \cap P$ which is also a vertex of P. Then $a \in L(X, P)$. Hence $S^{n-1} = \bigcup_{X \in \operatorname{vert} P} L(X, P)$. Since $L(X_1, P) \cap L(X_2, P)$ is a set of measure 0 for $X_1 \neq X_2$ we get the result.

Identities of Sommerville and McMullen

1.8. There are three basic identities for every convex polytope P and every face F of P:

$$\sum_{F \leqslant G \leqslant P} (-1)^{\dim P - \dim G} \alpha(G, P) = \alpha(F, P) \qquad (1.8.1)$$

$$\sum_{F \leqslant G \leqslant P} (-1)^{\dim G - \dim F} \beta(F, G) = \beta(F, P) \tag{1.8.2}$$

$$\sum_{F \leqslant G \leqslant P} \alpha(F, G)\, \beta(G, P) = 1. \tag{1.8.3}$$

D. M. Y. Sommerville [So] published an (incorrect) proof of (1.8.1) extending a relation of J. P. Gram [Gr] for 3-dimensional convex polytopes. For a correct proof of (1.8.1) see [PS] (for Gram's theorem we refer the reader to [Grü, Theorem 14.1.1; Sh]). Equation (1.8.2) is the dual version of Sommerville's Equation (1.8.1). It can be obtained by applying (1.8.1) to the polar convex polytope P^* of P (see [MS, 2.2]). Equation (1.8.3) has been proved by P. McMullen in [Mc].

Cell Complexes

1.9. A *cell complex* K is a finite collection of cells in some $\mathbf{R}^d$ satisfying

(1) If $C \in K$ and $B < C$ then $B \in K$.

(2) If $B, C \in K$ then $B \cap C$ is a face of both B and C.

Define the *underlying polyhedron* $|K|$ to be the union of all cells of K. A cell complex K is a *simplicial complex* if each cell of K is a simplex.

1.10. $|K| = \bigsqcup_{C \in K} C^\circ$.

Proof. By 1.5,

$$|K| = \bigcup_{C \in K} C = \bigcup_{C \in K} \bigcup_{F \leqslant C} F^\circ = \bigcup_{C \in K} C^\circ.$$

Assume $C \neq D$ then $C \cap D$ is a proper face of both C and D. Let $C \cap D < C$, then $C \cap D \subseteq \dot{C}$ and therefore $C \cap D \cap C^\circ = \varnothing$ by 1.5 and $C^\circ \cap D^\circ \subseteq C \cap D \cap C^\circ = \varnothing$.

Subdivision

1.11. A cell complex L is a *subdivision* of the cell complex K, written $L \lhd K$, if each cell of L is contained in a cell of K and if $|L| = |K|$. Every subdivision $L \lhd K$ defines a map $\varphi\colon L \to K$ by

$$\varphi(C) := \bigcap_{\substack{B \in K \\ C \subseteq B}} B,$$

i.e., $\varphi(B)$ is the smallest cell of K containing C. φ is an isotone map from the poset $(L, \leqslant)$ into the poset $(K, \leqslant)$.

1.12. Lemma. (1) $\varphi(C) = D$ *iff* $C^\circ \subseteq D^\circ$.

(2) $\varphi(C)^\circ \cap D \neq \varnothing$ *iff* $\varphi(C) \leqslant D$ *iff* $C \subseteq D$.

(3) *If* $D \in K$ *then* $D = \bigsqcup_{C \subseteq D} C^\circ$.

(4) $C = D$ *iff* $C \leqslant D$.

(5) $C^\circ \cap D \neq \varnothing$ *iff* $C \leqslant D$.

Proof. (1) Assume $\varphi(C) = D$ and $x \in C^\circ$. Since $C \subseteq D$ we get $\langle C \rangle \subseteq \langle D \rangle$. Let U be a neighborhood of x in $\langle C \rangle$ with $U \subseteq C$. Let H^+ be a closed half-space with $D \subseteq H^+$. Then $U \subseteq H^+ \cap \langle C \rangle$ which is equal to $\langle C \rangle$ or a closed half-space of $\langle C \rangle$. Since U is open U is contained in the interior of $H^+ \cap \langle C \rangle$ which is equal to the intersection of the interior $(H^+)^\circ$ of H^+ with $\langle C \rangle$ if not $\langle C \rangle \subseteq H$. By 1.3, D is the intersection of a finite number of closed half-spaces H_i^+ belonging to supporting hyperplanes H_i of D. As we have proved, $\langle C \rangle \subseteq H_i$ or $U \subseteq H_i^{+\circ}$ for all i. If $\langle C \rangle \subseteq H_i$ then $C \subseteq D \cap H_i$ which is a proper face of D in contradiction to $\varphi(C) = D$. Hence $U \subseteq H_i^{+\circ}$ for all i, $U \subseteq \bigcap H_i^{+\circ} = D^\circ$ and $C^\circ \subseteq D^\circ$.

On the other side $C^\circ \subseteq D^\circ$ implies $C \subseteq D$, hence $\varphi(C) \leqslant D$. Assume $\varphi(C) < D$, then by 1.5, $\varphi(C) \cap D^\circ = \varnothing$. Hence $C^\circ = C^\circ \cap D^\circ \subseteq \varphi(C) \cap D^\circ = \varnothing$, a contradiction.

(2) If not $\varphi(C) \leqslant D$ then $\varphi(C) \cap D$ is a proper face of $\varphi(C)$. Hence $\varphi(U) \cap D \subseteq \varphi(C)$ and therefore by 1.5, $\varphi(C) \cap D \cap \varphi(C)^\circ = \varnothing$, $\varphi(C)^\circ \cap D = \varnothing$. Hence $\varphi(C)^\circ \cap D \neq \varnothing$ implies $\varphi(C) \leqslant D$. The other direction is straightforward.

(3) Let $x \in D$, then $x \in |K| = |L| = \bigsqcup_{C \in L} C^\circ$ by 1.10. Assume $x \in C^\circ$, $C \in L$ then $x \in C^\circ \cap D \subseteq \varphi(C)^\circ \cap D$, i.e., $\varphi(C)^\circ \cap \neq \varnothing$ and this implies $C \subseteq D$ by (2). Hence $D \subseteq \bigcup_{C \subseteq D, C \in L} C^\circ$. The other direction is obvious.

(4) and (5) are straightforward implications of (1) and (2), respectively, if we set $L = K$ and $\varphi = 1_K$.

1.13. Proposition. *Let* $L \lhd K$ *be a subdivision and* $\varphi\colon L \to K$ *be the corresponding mapping. Then* (1)

$$\sum_{\substack{C \in L \\ \varphi(C) = D \\ \dim C = \dim D}} \operatorname{vol}(C) = \operatorname{vol}(D).$$

(2) *If* $B \in L$, $D \in K$, *and* $\varphi(B) \leqslant D$ (*i.e.*, $B \subseteq D$) *then*

$$\sum_{\substack{B \subseteq C \in L \\ \varphi(C) = D \\ \dim C = \dim D}} \alpha(B, C) = \alpha(\varphi(B), D).$$

Proof. (1) From 1.12(3) follows $\operatorname{vol}(D) = \sum_{C \subseteq D} \operatorname{vol}(C^\circ)$. Let $C \subseteq D$. If $\varphi(C) \neq D$ then $\varphi(C)$ is a proper face of D. Hence $\dim C \leqslant \dim \varphi(C) < \dim D$, hence $\operatorname{vol}(C) = \operatorname{vol}(C^\circ) = 0$ and this implies (1).

(2) We are using the second definition of the interior angle in 1.6. Let $z \in B^\circ \subseteq \varphi(B^\circ)$ and let h be any ray through z which points into D.

$$h \cap D = h \cap \bigcup_{\substack{C \in L \\ C \subseteq D}} C^\circ = \bigcup_{\substack{C \in L \\ C \subseteq D}} h \cap C^\circ.$$

Since $h \cap C^\circ$ is either empty or a point or an open arc there is an open arc $h \cap C^\circ$ which is nearest to z, i.e., z is contained in the topological closure of $h \cap C^\circ$. Therefore $z \in C$. Hence $z \in C \cap B^\circ$. This implies $B \leqslant C$ by 1.12(5) and $h \in \mathbf{C}(B, C)$. Hence

$$\mathbf{C}(\varphi(B), D) = \bigsqcup_{\substack{C \in L \\ B \leqslant C \leqslant D}} \mathbf{C}(B, C).$$

Let D be an n-cell then

$$S^{n-1} \cap \mathbf{C}(\varphi(B), D) = \bigsqcup_{\substack{C \in L \\ B \leqslant C \leqslant D}} (S^{n-1} \cap \mathbf{C}(B, C))$$

whence

$$\begin{aligned} \alpha(\varphi(B), D) &= \frac{1}{O_{n-1}} \sum_{\substack{C \in L \\ B \leqslant C \subseteq D}} \operatorname{vol}(S^{n-1} \cap \mathbf{C}(B, C)) \\ &= \frac{1}{O_{n-1}} \sum_{\substack{C \in L \\ B \subseteq C,\, \varphi(C) = D \\ \dim C = n}} \operatorname{vol}(S^{n-1} \cap \mathbf{C}(B, C)) \\ &= \sum_{\substack{C \in L \\ B \subseteq C,\, \varphi(C) = D \\ \dim C = n}} \alpha(B, C), \end{aligned}$$

because vol is the $(n-1)$-dimensional content and therefore

$$\operatorname{vol}(S^{n-1} \cap \mathbf{C}(B, C)) = 0$$

if $\dim C < n$ or if $\varphi(C) < D$ (which also implies $\dim C \leqslant \dim \varphi(C) < \dim D = n$).

1.14. Proposition. *A cell complex can be subdivided to a simplicial complex without introducing any new vertices.*

(Proof in [RS, 2.9].)

Products

1.15. If K and L are cell complexes then their product

$$K \times L := \{A \times B \mid A \in K,\ B \in L\}$$

(where $A \times B$ is the cartesian product of the cells A and B) is a cell complex. $C \times D$ is a face of $A \times B$ if $C \leqslant A$ and $D \leqslant B$. If $\alpha(C, A)$ and $\alpha(D, B)$ are the interior angles of A at C and of B at D, respectively, then

$$\alpha(C \times D, A \times B) = \alpha(C, A)\, \alpha(D, B).$$

A proof of this fact can be found in [Mc, Lemma 2].

2. Posets

2.1. For basic notations of the theory of partially ordered sets (posets for short) we refer the reader to the fundamental paper [Ro1] of G.-C. Rota.

A finite poset P is said to be *homogeneous* if all maximal chains have the same length. A homogeneous poset satisfies the *Jordan–Dedekind condition*: if x and y are two elements and if $x < y$ then $P_{\leqslant x} := \{z \mid z \leqslant x\}$, $P_{\geqslant x} := \{z \mid x \leqslant z\}$, and $[x, y] := \{z \mid x \leqslant z \leqslant y\}$ are homogeneous.

The symbol "$\prec$" denotes the *covering relation*: $x \prec y$ if $x < y$ and if $x < z \leqslant y$ implies $z = y$. If P is a finite homogeneous poset then a *rank function* $r\colon P \to \mathbf{N}$ can be defined as follows:

(i) If P has a least element 0 then we define $r(0) := 0$, otherwise we define $r(x) := 1$ for all minimal elements x.

(ii) If $x \prec y$ then $r(y) := r(x) + 1$.

A *finite simplicial complex* K is by definition a nonempty family of non-empty subsets called *simplexes* of a set $\{v\}$ of vertices such that

(i) any set consisting of exactly one vertex is a simplex,

(ii) any nonempty subset of a simplex is a simplex.

For details we refer the reader to [Sp]. A simplex σ which is contained in a simplex τ is called a *face* of τ. The *dimension* of a simplex σ, dim σ, is the maximum of the dimensions of all simplexes of K. The maximal simplexes, i.e., those simplexes which are maximal under inclusion, are called *facets*. K is said to be *homogeneously n-dimensional* if every simplex is a face of an n-dimensional simplex. So in this case all facets are n-dimensional.

Every finite simplicial complex K defines a finite poset $(K, <)$ the elements of which are the simplexes of K and these are partially ordered by inclusion. If K is homogeneously n-dimensional then the corresponding poset is homogeneous of rank $n+1$. Its rank function r satisfies obviously the following condition: $r(\sigma) = \dim \sigma + 1$.

The Incidence Algebra

2.2. Let P be a locally finite poset, i.e., a poset for which all segments $[x, y]$ are finite. Let $\mathscr{A}(P)$ be the *incidence algebra* of P consisting of all real valued functions (called *incidence functions*) $f: P^2 \to \mathbf{R}$ with $f(x, y) \neq 0$ only if $x \leqslant y$. The sum of two functions and the product of a function by a real number are defined in the obvious way, and an associative multiplication is defined by

$$(fg)(x, y) := \sum_{w \in [x, y]} f(x, z)\, g(z, y).$$

The identity element of $\mathscr{A}(P)$ is the Kronecker delta δ defined by

$$\delta(x, y) := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

It can be proved that an element $f \in \mathscr{A}(P)$ is a unit in this algebra, i.e., it has an inverse, iff $f(x, x) \neq 0$ for all $x \in P$.

2.3. There are two basic incidence functions: the zeta function $\zeta(x, y)$ defined by

$$\zeta(x, y) := \begin{cases} 1 & \text{if } x \leqslant y \\ 0 & \text{otherwise} \end{cases}$$

and the function $\eta = \zeta - \delta$. ζ is invertible ($\zeta(x, x) = 1$ for all x) and the inverse ζ^{-1} is called the Möbius function μ. From $\mu\zeta = \delta$ we get the following recursive definition of μ:

(i) $\mu(x, x) = 1$ for all $x \in P$,

(ii) $\mu(x, y) = -\sum_{z \in [x, y)} \mu(x, z)$ if $x < y$,

(iii) $\mu(x, y) = 0$ if $x \nleqslant y$,

where $[x, y) = \{z \mid x \leqslant z < y\}$ is the left closed right open interval.

2.4. Let K be a cell complex. K with the "face of" relation $\leqslant$ can be considered as a poset with two distinguished incidence functions: the *interior angle function* α and the *exterior angle function* β. $\alpha(F, P)$ and $\beta(F, P)$ are defined for $F \leqslant P$ as in 1.6 and for $F \nleqslant P$ we set $\alpha(F, P) = \beta(F, P) = 0$. P. McMullen pointed out that the identity (1.8.3) can be expressed by $\alpha\beta = \zeta$. Moreover define $\bar{\alpha}$ and $\bar{\beta}$ by

$$\bar{\alpha}(F, P) = (-1)^{\dim P - \dim F} \alpha(F, P) \tag{2.4.1}$$

$$\bar{\beta}(F, P) = (-1)^{\dim P - \dim F} \beta(F, P). \tag{2.4.2}$$

Then Eqs. (1.8.1) and (1.8.2) become the following identities, respectively:

$$\zeta\bar{\alpha} = \alpha \qquad \text{and} \qquad \bar{\beta}\zeta = \beta.$$

Hence we have got the following proposition:

2.5. PROPOSITION. *For every cell complex K there are three basic identities*:

(*Sommerville-identities*) $\zeta\bar{\alpha} = \alpha$ *and* $\bar{\beta}\zeta = \beta$

(*McMullen-identity*) $\alpha\beta = \zeta$.

Hence β, $\bar{\alpha}$, and $\bar{\beta}$ can be expressed by α via the equations

$$\beta = \alpha^{-1}\zeta,$$

$$\bar{\alpha} = \mu\alpha = \beta^{-1},$$

$$\bar{\beta} = \beta\mu = \alpha^{-1},$$

2.6. Let f be an element of $\mathscr{A}(P)$ satisfying the property

$$f(x, x) = 0 \qquad \text{for all} \quad x \in P. \tag{2.6.1}$$

Fact. If $f^n(x, y) \neq 0$ then there is a chain of length n,

$$x = x_0 < x_1 < \cdots < x_n = y,$$

connecting x with y.

Proof. By induction on n: For $n = 1$ this is exactly the property assumed for f. Assume $f^{n+1}(x, y) \neq 0$, i.e.,

$$f^{n+1}(x, y) = \sum_{x \leqslant z \leqslant y} f^n(x, z) f(z, y) \neq 0.$$

Hence there is at least one z with $f^n(x, z) \neq 0$ and $f(z, y) \neq 0$ whereas it follows that $z < y$ and by induction there is a chain $x = x_0 < x_1 < \cdots < x_n = z$. Hence $x = x_0 < x_1 < \cdots < x_n = z < x_{n+1} = y$ is the desired chain of length $n + 1$ connecting x with y.

2.7. COROLLARY. *If f satisfies* (2.6.1) *then $\delta - f$ is invertible and for all $x, y \in P$ there is an n with $f^t(x, y) = 0$ for $t > n$, $f^n(x, y) \neq 0$ and*

$$(\delta - f)^{-1}(x, y) = \delta(x, y) + f(x, y) + \cdots + f^n(x, y).$$

Proof. n is equal to the maximum of all natural numbers such that there exists a chain of length n connecting x with y. Then $(\delta+f+\cdots+f^n)(\delta-f)=\delta-f^{n+1}$ whence

$$\begin{aligned}(\delta+f+\cdots+f^n)(x,y)&=((\delta-f^{n+1})(\delta-f)^{-1})(x,y)\\&=\sum_{x\leqslant z\leqslant y}(\delta-f^{n+1})(x,z)(\delta-f)^{-1}(z,y)\\&=(\delta-f)^{-1}(x,y).\end{aligned}$$

2.8. Formally we write

$$(\delta-f)^{-1}=\sum_{i=1}^{\infty}f^i.$$

If g is an element of $\mathscr{A}(P)$ with $g(x,x)=1$ for all $x\in P$, $f=\delta-g$ satisfies condition (2.6.1) and we get

$$g^{-1}=(\delta-f)^{-1}=\sum_{i=0}^{\infty}f^i=\sum_{i=0}^{\infty}(\delta-g)^i.$$

2.9. Let $\Delta_n(P)$ be the set of all subsets $\{x_0,x_1,...,x_n\}$ of P such that $x_0<x_1<\cdots<x_n$. $\Delta(P)=\bigcup_{n=0}^{\infty}\Delta(P)$ is a simplicial complex, the vertices of which are the elements of P. If K is a simplicial complex then $K':=\Delta(K)$ (K to be considered as a poset) is called the *barycentric subdivision of K*.

Let P be an arbitrary finite poset. In this case $\Delta(P)$ is finite. The alternating sum $\sum_{n=0}^{\infty}(-1)^n\,\#(\Delta_n(P))$ is the *Euler–Poincaré-characteristic* $\chi(\Delta(P))$ of the simplicial complex $\Delta(P)$, which will be denoted also by $\chi(P)$ (remark that $\chi(\varnothing)=0$). Let $x,y\in P$ and $x<y$. (x,y) denotes the open interval of all $z\in P$ with $x<z<y$. It is easy to verify that for $x<y$, $\#(\Delta_n(x,y))=\eta^{n+2}(x,y)$. This yields the following

2.10. THEOREM (Hall). $\chi(x,y)=1+\mu(x,y)$.

Proof.

$$\mu(x,y)=\zeta^{-1}(x,y)=(\delta-(-\eta))^{-1}=\sum_{n=0}^{\infty}(-1)^n\eta^n(x,y)$$

$$=-1+\sum_{n=0}^{\infty}(-1)^n\,\#\,\Delta_n(x,y)=-1+\chi(x,y).$$

2.11. Let C be an n-cell and let $P=\mathscr{F}(C)$ be the lattice of all faces of C. Let B be an m-cell which is a face of C, i.e., $B\leqslant C$. Let C^* be the polar convex polytope of C in the sense of [MS, Sect. 2.2]. The lattice $\mathscr{F}(C^*)$ is dual to $\mathscr{F}(C)$, i.e., $\mathscr{F}(C^*)$ is the dual lattice $(\mathscr{F}(C))^*$ of $\mathscr{F}(C)$. Let

$$\begin{aligned}(\mathscr{F}(C))^*&\to\mathscr{F}(C^*)\\B&\mapsto B^\circ\end{aligned}$$

be the canonical isomorphism. The closed interval $[B, C]$ is dual to $\mathscr{F}(B^\circ)$ because

$$[B, C]^* \cong [C^\circ, B^\circ] \cong [\varnothing, B^\circ] \cong \mathscr{F}(B^\circ).$$

Hence (B, C) is dual to $\mathscr{F}(B^\circ) - \{B^\circ, \varnothing\}$. Since B° is an $(n-m-1)$-cell (B, C) is dual to the poset of all faces of an $(n-m-1)$-cell which are different from $\varnothing$ and contained in its boundary. We now need the following lemma:

2.12. Lemma. *Let C be an n-polytope and let $\mathscr{F}^\bullet(C)$ be the cell complex $\mathscr{F}(C) - \{C, \varnothing\}$ of all faces of C which are different from $\varnothing$ and contained in the boundary of C. Then*

$$\chi(\mathscr{F}^\bullet(C)) = 1 - (-1)^n.$$

Proof. Let F be any face of C and let $z(F)$ be the centroid of F. Let $\sigma = \{F_0 < F_1 < \cdots < F_n < F\}$ be any simplex of $\Delta(\mathscr{F}^\bullet(C))$. Let $z(\sigma) := z(F_0)\, z(F_1) \cdots z(F_n)$ be the simplex which is the convex hull of $\{z(F_0), z(F_1), ..., z(F_n)\}$. The set of all $z(\sigma)$, $\sigma \in \Delta(\mathscr{F}^\bullet(C))$, is a subdivision of the cell complex $\mathscr{F}^\bullet(C)$ which is a cell complex with underlying polyhedron $C^\bullet$ which is an $(n-1)$-sphere. Hence $\Delta(\mathscr{F}^\bullet(C))$ is the simplicial complex of a triangulation of the $(n-1)$-sphere $C^\bullet$ which yields

$$\chi(\Delta(\mathscr{F}^\bullet(C))) = 1 + (-1)^{n-1} = 1 - (-1)^n,$$

the result.

2.13. Corollary. *Let C be an n-cell, B an m-cell, and $B \leqslant C$. Then*

$$\mu(B, C) = (-1)^{n-m}.$$

Proof. As we proved in 2.11, the interval (B, C) is dual to $\mathscr{F}^\bullet(B^\circ)$ and B° is an $(n-m-1)$-cell. Hence by 2.10 and 2.12

$$\begin{aligned}\mu(B, C) &= \chi(B, C) - 1 = \chi(\mathscr{F}^\bullet(B^\circ)^*) - 1 \\ &= \chi(\mathscr{F}^\bullet(B^\circ)) - 1 = 1 - (-1)^{n-m-1} - 1 = (-1)^{n-m}.\end{aligned}$$

3. Chern–Gauss–Bonnet Densities

3.1. J. Cheeger, W. Müller, and R. Schrader defined the Chern–Gauss–Bonnet density for simplicial complexes. As a matter of fact, this definition can be generalized to arbitrary cell complexes in a straightforward way. Let

K be a cell complex and let C be a cell. The *Chern–Gauss–Bonnet density* $r(C)$ of C (denoted in [CMS] by $P_\chi(C^\perp(\sigma))$ for a simplex σ) is defined by

$$r(C) := \sum_{C \leqslant D} (-1)^{\dim D - \dim C} \beta(C, D), \tag{3.1.1}$$

i.e., it is the alternating sum of the exterior angles at C of all cells of K having C as a face. In case that all maximal cells of K have the same dimension n which is called the dimension of K we define the *Lipschitz–Killing curvature* R^j of K by

$$R^j(K) := \sum_{\dim C = n-j} \operatorname{vol}(C)\, r(C). \tag{3.1.2}$$

This definition generalizes slightly the definition of [CMS] for simplicial complexes to arbitrary cell complexes. We will prove in Section 6 that this definition naturally extends the definition of [CMS] because the Lipschitz–Killing curvatures of triangulations of K coincide with the Lipschitz–Killing curvatures of K. Consider K as a poset as in 2.3 and add to this poset a maximal element $\hat{1}$ with $C < \hat{1}$ for all $C \in K$. Using the notations of 2.2 and 2.3, formula (3.1.1) can be written as

$$r(C) = \sum_{C \leqslant D} \bar{\beta}(C, D) = \sum_{C \leqslant D \leqslant \hat{1}} \bar{\beta}(C, D)\, \eta(D, \hat{1}) = \bar{\beta}\eta(C, \hat{1}) \tag{3.1.3}$$

(remark that in the last sum the term $\bar{\beta}(C, \hat{1})\, \eta(\hat{1}, \hat{1})$ is not strictly defined because $\bar{\beta}(C, \hat{1})$ is not defined; but this does not really matter because $\eta(\hat{1}, \hat{1}) = 0$). Using Proposition 2.4 we can in turn reformulate (3.1.3) to

$$r(C) = \alpha^{-1}\eta(C, \hat{1}). \tag{3.1.4}$$

This formula is the motivation for the following general definition:

3.2. Definition. Let P be a poset and let α be an incidence function of P with $\alpha(x, x) = 1$ for all $x \in P$. Define the incidence function r_α by

$$r_\alpha := \alpha^{-1}\eta.$$

Let $P_{\hat{1}}$ be the extension of P by an element $\hat{1}$ which is greater than all $x \in P$. Let $\bar{\alpha}$ be an arbitrary extension of α to $P_{\hat{1}}$ with $\bar{\alpha}(\hat{1}, \hat{1}) = 1$. Then $\bar{\alpha}^{-1}$ exists and we get: $r_{\bar{\alpha}}$ depends only on α and not on the choice of the extension $\bar{\alpha}$. For the proof of this statement remark first that for $x, y \in P$, $\bar{\alpha}^{-1}(x, y) = \alpha^{-1}(x, y)$. So $\bar{\alpha}^{-1}$ is an extension of α^{-1} to $P_{\hat{1}}$. We will prove indeed a sharper result.

3.2.1. *If φ is any incidence function (not necessarily invertible) of $P_{\hat{1}}$ which is an extension of α^{-1}, then*

$$\varphi\eta(x, y) = \sum_{x \leqslant z \leqslant y} \alpha^{-1}(x, z).$$

Remark that for all $y \in P_{\hat{1}}$ holds that: if $z < y$ then $z \in P$. So $\varphi\eta$ depends only on α.

Proof.

$$\begin{aligned}\varphi\eta(x, y) &= \sum_{x \leqslant z \leqslant y} \varphi(x, z)\,\eta(z, y)\\ &= \sum_{x \leqslant z < y} \varphi(x, z) = \sum_{x \leqslant z < y} \alpha^{-1}(x, z).\end{aligned}$$

Thus we write also r_α instead of $r_{\bar{\alpha}}$. So r_α is in the following an incidence function on $P_{\hat{1}}$ which is called the *Chern–Gauss–Bonnet density* of α or, for short, the α-density. Moreover we define for $x \in P$,

$$r_\alpha(x) := r_\alpha(x, \hat{1}).$$

In the following α will be fixed and we write r instead of r_α.

3.3. Proposition. *The α-density r satisfies the following conditions*:

(i) $r(x, y) = \sum_{x \leqslant z < y} \alpha^{-1}(x, z)$, $r(x) = \sum_{x \leqslant z} \alpha^{-1}(x, z)$. *Especially*: *if not $x < y$ then $r(x, y) = 0$.*

(ii) *If x is a maximal in P then $r(x) = 1$.*

(iii) $r(x) = 1 - \sum_{x < y} \alpha(x, y)\, r(y)$.

Proof. Part (i) is only the explicit formulation of $r = \alpha^{-1}\eta$ and (ii) is a straightforward consequence of (i). Part (iii) results from the following lemma:

3.4. Lemma. *For all $x \in P$ it holds that*

$$\sum_{x \leqslant y} \alpha(x, y)\, r(y) = 1.$$

Proof. $r = \alpha^{-1}\eta$ implies $\eta = \alpha r$, $\zeta = \alpha r + \delta$, $1 = \zeta(x, \hat{1}) = \alpha r(x, \hat{1}) = \sum_{x \leqslant y} \alpha(x, y)\, r(y, \hat{1})$.

3.5. Corollary. *For all $x \in P$ it holds that*

$$r(x) = 1 - \sum_{\substack{x < y\\ y \text{ not maximal}}} \alpha(x, y)\, r(y) - \sum_{\substack{x < y\\ y \text{ maximal}}} \alpha(x, y).$$

3.6. Proposition. $r = \sum_{n=0}^{\infty} (\delta - \alpha)^n\, \eta$, *i.e.*,

$$r(x) = 1 + \sum (-1)^n\, \alpha(x_0, x_1)\, \alpha(x_1, x_2) \cdots \alpha(x_{n-1}, x_n),$$

where the sum is taken over all chains

$$x = x_0 < x_1 < \cdots < x_n$$

of elements in P.

Proof. This is a straightforward consequence from $r = \alpha^{-1}\eta$ and 2.5.

3.7. THEOREM. *Define* $\beta = \alpha^{-1}\zeta$, $\gamma = \frac{1}{2}(\alpha + \beta^{-1})$. *Then* $\beta(x, x) = \gamma(x, x) = 1$ *for all* $x \in P$, *i.e.,* β *and* γ *are invertible incidence functions. Moreover*:

$$r = \tfrac{1}{2}\gamma^{-1}(\zeta - \mu).$$

Proof. $\beta(x, x) = \alpha^{-1}(x, x)\,\zeta(x, x) = 1$, $\gamma(x, x) = \frac{1}{2}(\alpha(x, x) + \beta^{-1}(x, x)) = 1$. Hence γ is invertible. The further proof will be divided into two steps.

(i) $\gamma^{-1} = 2\alpha^{-1}(\delta + \mu)^{-1}$ since $\gamma^{-1} = 2(\alpha + \beta^{-1})^{-1} = 2(\alpha + \mu\alpha)^{-1} = 2((\delta + \mu)\alpha)^{-1} = 2\alpha^{-1}(\delta + \mu)^{-1}$. From this results $\alpha^{-1} = \frac{1}{2}\gamma^{-1}(\delta + \mu)$.

(ii) $r = \alpha^{-1}\eta = \alpha^{-1}(\zeta - \delta) = \frac{1}{2}\gamma^{-1}(\delta + \mu)(\zeta - \delta) = \frac{1}{2}\gamma^{-1}(\zeta - \mu)$.

3.8. COROLLARY. $r = \frac{1}{2}\sum_{n=0}^{\infty} (\delta - \gamma)^n\,(\zeta - \mu)$.

Proof. Proposition 2.5 ad Theorem 3.7 yield the result.

3.9. COROLLARY.

$$r(x) = \chi^{\perp}(x) + \sum (-1)^n\,\gamma(x_0, x_1)\,\gamma(x_1, x_2)\cdots\gamma(x_{n-1}, x_n)\,\chi^{\perp}(x_n),$$

where the sum is taken over all chains $x = x_0 < x_1 < \cdots < x_n$ *of elements in P and* $\chi^{\perp}(x)$ *is defined by*

$$\chi^{\perp}(x) := 1 - \chi(P_{>x}).$$

3.10. Let K be a cell complex. By 2.4, $\beta^{-1} = \bar{\alpha}$ and therefore $\gamma = \frac{1}{2}(\alpha + \bar{\alpha})$, $\gamma(C, D) = (1 + (-1)^{\dim D - \dim C})\,\alpha(C, D)$ by 2.3. Hence $\gamma(C, D) \neq 0$ only if $\dim D - \dim C$ is even and in this case $\gamma(C, D) = \alpha(C, D)$. Hence

$$r(c) = \chi^{\perp}(C) + \sum (-1)^n\,\alpha(C_0, C_1)\,\alpha(C_1, C_2)\cdots\alpha(C_{n-1}, C_n)\,\chi^{\perp}(C_n),$$

where the sum is taken over all chains $C = C_0 < C_1 < \cdots < C_n$ with $\dim C_{i+1} - \dim C_i$ even for all $i = 0, 1, \ldots, n-1$. Moreover: if K is a simplicial complex, then $K_{>\sigma}$ is, for any simplex σ of K isomorphic to $L(\sigma)$, the combinatorial link of σ in K, i.e., the set of all simplexes τ with $\sigma \cap \tau = \varnothing$ and $\sigma \cup \tau$ is a simplex in K. Hence

$$\chi^{\perp}(\sigma) = 1 - \tfrac{1}{2}\chi(L(\sigma)).$$

This gives the following theorem of J. Cheeger [Ch] which Cheeger proved by heat equation methods; the proof in [CMS] uses also an analytical argumentation whereas our proof was basically combinatorial in nature: All properties follow essentially from the relations of Gram–Sommerville and McMullen.

3.11. THEOREM. *Let K be a simplicial complex and σ be a simplex of K. Then*

$$r(\sigma) = \chi^{\perp}(\sigma) + \sum (-1)^n \alpha(\sigma_0, \sigma_1)\, \alpha(\sigma_1, \sigma_2) \cdots \alpha(\sigma_{n-1}, \sigma_n)\, \chi^{\perp}(\sigma_n),$$

where $\chi^{\perp}(\sigma) = 1 - \frac{1}{2}\chi(L(\sigma))$ *and the sum is taken over all chains* $\sigma = \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n$ *with* $\dim \sigma_{i+1} - \dim \sigma_i$ *even for all* $i = 0, 1, ..., n-1$.

Consider the following simple application of this theorem: If x is a point in a piecewise flat manifold of dimension two, we have $\chi(L(x)) = 0$. From this results $\chi^{\perp}(x) = 1$ and therefore

$$r(x) = 1\left(1 - \sum \alpha(x, \sigma)\right) = 1 - \sum \alpha(x, \sigma),$$

where the sum is taken over all simplexes σ of dimension two with $x \in \sigma$. $r(x)$ is the angle defect at x.

4. LIPSCHITZ–KILLING CURVATURES

Let K be an arbitrary cell complex. The interior angle function $\alpha \in \mathscr{A}(K)$ satisfies the condition $\alpha(C, C) = 1$ for all cells C. On the other side every cell has a volume $\mathrm{vol}(C)$. The triple $(K, \alpha, \mathrm{vol})$ is a model of what we call in the sequel *angular partially ordered set.*

Angular Posets

4.1. An *angular partially ordered set*, angular poset for short, is a triple (P, α, v), consisting of a homogeneous poset P (see 2.1), an incidence function α of P with $\alpha(x, x) = 1$ for all $x \in P$, and a function $v: P \to \mathbf{R}$ with $v(x) \neq 0$ for all $x \in P$. α has an inverse α^{-1} in $\mathscr{A}(P)$. Define

$$\beta = \alpha^{-1}\zeta \qquad \text{and} \qquad \gamma = \tfrac{1}{2}(\alpha + \beta^{-1}).$$

Then $\alpha\beta = \zeta$, $\mu\alpha\beta = \alpha\beta\mu = \delta$ and this implies

$$\beta^{-1} = \mu\alpha, \qquad \alpha^{-1} = \beta\mu, \qquad \gamma = \tfrac{1}{2}(\delta + \mu)\alpha.$$

Remark that for all $x \in P$ it holds that

$$\beta(x, x) = \beta^{-1}(x, x) = \alpha^{-1}(x, x) = \gamma(x, x) = \gamma^{-1}(x, x) = 1.$$

ν defines an incidence function (also denoted by ν) of the incidence algebra $\mathscr{A}(P)$ in the following way:

$$\nu(x, y) := \begin{cases} \nu(x) & \text{if } \ x = y \\ 0 & \text{otherwise.} \end{cases}$$

Let $f(x, y)$ be an arbitrary element of $\mathscr{A}(P)$. Then

$$(f\nu)(x, y) = \sum_z f(x, z)\, \nu(z, y) = f(x, y)\, \nu(y)$$

$$(\nu f)(x, y) = \sum_z (x, z)\, f(z, y) = \nu(x)\, f(x, y).$$

Hence

$$\nu^{-1}(x, y) = \begin{cases} \nu(x)^{-1} & \text{if } \ x = y \\ 0 & \text{otherwise.} \end{cases}$$

4.2. In every angular poset a *(Chern–Gauss–Bonnet) density* $r = r_\alpha = \alpha^{-1}\eta$ is defined (see 3.2). r is called the *density* of $P = (P, \alpha, \nu)$ and is denoted by r^P. Let j be any integer. The jth *curvature* $R_j(P)$ of P is defined by

$$R_j(P) := \sum_{\mathrm{rk}(x) = j} \nu(x)\, r(x).$$

Let P be of rank $\mathrm{rk}(P)$. The jth *Lipschitz–Killing curvature* $R^j(P)$ is defined by

$$R^j(P) := R_{\mathrm{rk}(P) - j}(P).$$

Remark that $R^j(P) = R_j(P) = 0$ for $j < 0$ and $j > \mathrm{rk}(P)$.

4.3. Let $P = (P, \alpha, \nu)$ be an angular poset and let Q be a descending subset of P, i.e., a subset Q of P which satisfies the condition: $x \leqslant y \in Q$ implies $x \in Q$. The triple $(Q, \alpha \mid Q^2, \nu \mid Q)$ is an angular poset which we denote by Q, too. Let r^P, r^Q be the densities of P, Q, respectively. In order to calculate $R_j(Q)$ we start with the following lemma:

4.4. Lemma. *Let $\eta_Q : P^2_{\hat{1}} \to \mathbf{R}$ be the function*

$$\eta_Q(x, y) := \begin{cases} 1 & \textit{if } \ x, y \in \hat{Q} \textit{ and } x < y \\ 0 & \textit{otherwise.} \end{cases}$$

Then

$$\alpha^{-1}\eta_Q(x, y) = \begin{cases} r^Q(x, y) & \textit{if } \ x, y \in \hat{Q} \textit{ and } x < y \\ 0 & \textit{otherwise.} \end{cases}$$

Proof.

$$\alpha^{-1}\eta_Q(x, y) = \sum_{x \leqslant z \leqslant y} \alpha^{-1}(x, z)\,\eta_Q(z, y)$$

$$= \begin{cases} \sum_{z \in Q, x \leqslant z < y} \alpha^{-1}(x, z) & \text{if } y \in \hat{Q} \\ 0 & \text{otherwise.} \end{cases}$$

If the sum is different from 0 then there is at least one $z \in Q$ with $x \leqslant z < y$. Since Q is descending this implies $x \in Q$. Therefore

$$\alpha^{-1}\eta_Q(x, y) = \begin{cases} \sum_{z \in Q, x \leqslant z < y} \alpha^{-1}(x, y) & \text{if } x \in Q,\ y \in \hat{Q},\ x < y \\ 0 & \text{otherwise.} \end{cases}$$

By 3.3(i) we get the result.

4.5. LEMMA. *Let S and T be two descending subsets of P. Then $S \cap T$ and $S \cup T$ are descending and*

$$\eta_{S \cup T} + \eta_{S \cap T} = \eta_S + \eta_T.$$

Proof. Let $\psi := \eta_{S \cup T} + \eta_{S \cap T} - \eta_S - \eta_T$. We have to prove that $\psi(x, y) = 0$ for all $x, y \in P$. This is obviously true, if not $x < y$. If $x < y$ we have to study essentially up to permutation of S and T the following cases:

x	y	η_S	η_T	$\eta_{S \cup T}$	$\eta_{S \cap T}$	ψ
$\in S \in T$	$\hat{1}$	1	1	1	1	0
$\in S \notin T$	$\hat{1}$	1	0	1	0	0
$\notin S \notin T$	$\hat{1}$	0	0	0	0	0
	$\neq \hat{1}$ and					
$\in S \in T$	$\in S \in T$	1	1	1	1	0
$\in S$	$\in S \notin T$	1	0	1	0	0
	$\notin S \notin T$	0	0	0	0	0

An empty field means that $\in$ and $\notin$ may be possible.

4.6. Let Q be a descending subset of P and let ρ_Q be the incidence function $\rho_Q := v\alpha^{-1}\eta_B$. From 4.5 it follows that for descending subsets S and T it holds that

$$\rho_{S \cup T} + \rho_{S \cap T} = \rho_S + \rho_T.$$

4.7. LEMMA. *Let Q be a descending subset of P. Then*

$$R_j(Q) = \sum_{\substack{x \in P \\ \mathrm{rk}(x) = j}} \rho_Q(x, \hat{1}).$$

Proof. By 4.4 we get

$$\rho_Q(x, \hat{1}) = v(x)(\alpha^{-1}\eta_Q)(x, \hat{1}) = \begin{cases} v(x)\, r^Q(x) & \text{if } x \in Q \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{\mathrm{rk}(x)=j} \rho_Q(x, \hat{1}) = \sum_{\substack{x \in Q \\ \mathrm{rk}(x)=j}} v(x)\, r^Q(x) = R_j(Q).$$

Now we have all the tools to prove the following theorem:

4.8. THEOREM. *Let S and T be two descending subsets of P. Then $S \cup T$ and $S \cap T$ are descending and for all natural numbers j it holds that*

$$R_j(S \cup T) + R_j(S \cap T) = R_j(S) + R_j(T).$$

Proof. By 4.7 and 4.4 we get

$$\begin{aligned} &R_j(S \cup T) + R_j(S \cap T) - R_j(S) - R_j(T) \\ &\quad = \sum_{\substack{x \in P \\ \mathrm{rk}(x)=j}} \rho_{S \cup T}(x, \hat{1}) + \rho_{S \cap T}(x, \hat{1}) - \rho_S(x, \hat{1}) - \rho_T(x, \hat{1}) = 0. \end{aligned}$$

4.9. COROLLARY. *Let P be a homogeneous angular poset and let S and T be two descending subsets of equal rank n. If $S \cap T$ is homogeneous of rank m then*

$$R^j(S \cup T) + R^{m+j-n}(S \cap T) = R^j(S) + R^j(T).$$

5. PRODUCTS

5.1. DEFINITION. Let $P = (P, \alpha_P, v_P)$ and $Q = (Q, \alpha_Q, v_Q)$ be two angular posets. The product $P \times Q = (P \times Q, \alpha_{P \times Q}, v_{P \times Q})$ is defined by

$$\alpha_{P \times Q}((x_1, y_1), (x_2, y_2)) = \alpha_P(x_1, x_2)\, \alpha_Q(y_1, y_2)$$

and

$$v_{P \times Q}(x, y) = v(x)\, v(y).$$

Remark that for all complexes K and L, the angular poset, corresponding to $K \times L$ (see the beginning of Section 4), is the product of the angular posets, corresponding to K and L (see 1.15). In the following we study the relationship between the Lipschitz–Killing curvatures of P, Q, and $P \times Q$. The basic result is the following relation between the Chern–Gauss–Bonnet densities.

5.2. PROPOSITION. *Let r^P, r^Q, $r^{P\times Q}$ be the densities of the angular posets P, Q, $P\times Q$, respectively. For all $x\in P$, $y\in Q$ it holds that*

$$r^{P\times Q}(x, y) = r^P(x)\, r^Q(y).$$

We are giving two proofs of this result. The first one is easy from the technical point of view but it does not give deeper insight into the nature of the result. The second is a little more difficult but it gives evidence why this result is true.

First Proof. This is a top-down induction in the poset $P\times Q$: If (x, y) is maximal in $P\times Q$ then x and y are maximal elements in P and Q, respectively. By 3.3(ii), $r^{P\times Q}(x, y) = r^P(x) = r^Q(y) = 1$, $r^{P\times Q}(x, y) = r^P(x)\, r^Q(y)$. Assume now that $r^{P\times Q}(x', y') = r^P(x')\, r^Q(y')$ for all (x', y') with $(x, y) < (x', y')$, i.e., for $x < x'$ or $y < y'$. Then using 3.3(iii) we get the result

$$\begin{aligned}
r^{P\times Q}(x, y) &= 1 - \sum_{(x,y)<(x',y')} \alpha_{P\times Q}((x, y)(x', y'))\, r^{P\times Q}(x', y')\\
&= 1 - \sum_{(x,y)<(x',y')} \alpha_P(x, x')\, \alpha_Q(y, y')\, r^P(x')\, r^Q(y')\\
&= 1 - \sum_{y<y'} \alpha_Q(y, y')\, r^P(x)\, r^Q(y) - \sum_{x<x'} \alpha_P(x, x')\, r^P(x')\, r^Q(y)\\
&\quad - \sum_{\substack{x<x'\\ y<y'}} \alpha_P(x, x')\, \alpha_Q(y, y')\, r^P(x')\, r^Q(y')\\
&= 1 - \sum_{y<y'} \alpha_Q(y, y') \left(1 - \sum_{x<x'} \alpha_P(x, x')\, r^P(x')\right) r^Q(y')\\
&\quad - \sum_{x<x'} \alpha_P(x, x')\, r^P(x') \left(1 - \sum_{y<y'} \alpha_Q(y, y')\right) r^Q(y')\\
&\quad - \sum_{\substack{x<x'\\ y<y'}} \alpha_P(x, x')\, \alpha_Q(y, y')\, r^P(x')\, r^Q(y')\\
&= 1 - \sum_{x<x'} \alpha_P(x, x')\, r^P(x') - \sum_{y<y'} \alpha_Q(y, y')\, r^Q(y')\\
&\quad + \sum_{\substack{x<x'\\ y<y'}} \alpha_P(x, x')\, \alpha_Q(y, y')\, r^P(x')\, r^Q(y')\\
&= \left(1 - \sum_{x<x'} \alpha_P(x, x')\, r^P(x')\right)\left(1 - \sum_{y<y'} \alpha_Q(y, y')\, r^Q(y')\right)\\
&= r^P(x)\, r^Q(y).
\end{aligned}$$

Second Proof. Let us first introduce some notations: If P is a poset, the η-function, introduced in 2.2, of P is denoted by η_P. If f and g are incidence functions of P, Q, respectively, then $f \times g$ is the incidence function of $P \times Q$, defined by $(f \times g)((x, y), (x', y')) := f(x, x')\, g(y, y')$. Obviously $(f_1 \times g_1)(f_2 \times g_2) = f_1 f_2 \times g_1 g_2$. If f is an incidence function on P and if S is a subposet of P then the restriction of f to S is denoted by $f \mid S$. Remark that for all posets P, Q it holds that

$$\eta_{P \times Q} = \eta_P \times \eta_Q + \delta_P \times \eta_Q + \eta_P \times \delta_Q \tag{5.2.1}$$

and therefore

$$\eta_{P_{\hat{1}} \times Q_{\hat{1}}} = \eta_{P_{\hat{1}}} \times \eta_{Q_{\hat{1}}} + \delta_{P_{\hat{1}}} \times \eta_{Q_{\hat{1}}} + \eta_{P_{\hat{1}}} \times \delta_{Q_{\hat{1}}}. \tag{5.2.2}$$

Furthermore

$$\eta_{P_{\hat{1}} \times Q_{\hat{1}}} \mid (P \times Q)_{\hat{1}} = \eta_{(P \times Q)_{\hat{1}}}, \tag{5.2.3}$$

where we identify $(P \times Q)_{\hat{1}}$ with the subposet $P \times Q \cup \{(\hat{1}, \hat{1})\}$ of $P_{\hat{1}} \times Q_{\hat{1}}$. Now let φ and ψ be incidence functions of $P_{\hat{1}}$ and $Q_{\hat{1}}$, which are extensions of α_P^{-1}, α_Q^{-1}, respectively. By 3.2.1, $r^P = \varphi \eta_{P_{\hat{1}}}$, $r^Q = \varphi \eta_{Q_{\hat{1}}}$. Obviously $\varphi \times \psi$ is an extension of $\alpha_P^{-1} \times \alpha_Q^{-1} = (\alpha_P \times \alpha_Q)^{-1} = (\alpha_{P \times Q})^{-1}$ to $P_{\hat{1}} \times Q_{\hat{1}}$. Hence $\varphi \times \psi \mid (P \times Q)_{\hat{1}}$ is an extension of $\alpha_{P \times Q}^{-1}$ to $(P \times Q)_{\hat{1}}$ and by 3.2.1, using (5.2.3), we get

$$\begin{aligned} r^{P \times Q} &= ((\varphi \times \psi) \mid (P \times Q)_{\hat{1}})\eta_{(P \times Q)_{\hat{1}}} \\ &= ((\varphi \times \psi) \mid (P \times Q)_{\hat{1}})(\eta_{P_{\hat{1}} \times Q_{\hat{1}}} \mid (P \times Q)_{\hat{1}}). \end{aligned} \tag{5.2.4}$$

By (5.2.2) the latter is the sum of three terms: The first term

$$t_1 := ((\varphi \times \psi) \mid (P \times Q)_{\hat{1}})((\eta_{P_{\hat{1}}} \times \eta_{P_{\hat{1}}}) \mid (P \times Q)_{\hat{1}})$$

satisfies

$$\begin{aligned} &t_1((x_1, y_1), (x_2, y_2)) \\ &\quad = \sum_{\substack{(x, y) \in (P \times Q)_{\hat{1}} \\ x_1 \leqslant x \leqslant x_2 \\ y_1 \leqslant y \leqslant y_2}} (\varphi \times \psi)((x_1, y_1), (x, y))(\eta_{P_{\hat{1}}} \times \eta_{Q_{\hat{1}}}((x, y), (x_2, y_2))) \\ &\quad = \sum_{\substack{x_1 \leqslant x < x_2 \\ y_1 \leqslant y < y_2}} \varphi(x_1, x)\, \psi(y_1, y) \\ &\quad = \sum_{\substack{x_1 \leqslant x < x_2 \\ y_1 \leqslant y < y_2}} \alpha_P^{-1}(x_1, x)\, \alpha_Q^{-1}(y_1, y) \\ &\quad = \left(\sum_{x_1 \leqslant x < x_2} \alpha_P^{-1}(x_1, x) \right) \left(\sum_{y_1 \leqslant y < y_2} \alpha_Q^{-1}(y_1, y) \right) \\ &\quad = r^P(x_1, x_2)\, r^Q(y_1, y_2), \end{aligned} \tag{5.2.5}$$

because $x < x_2$, $y < y_2$ implies $x \in P$, $y \in Q$, respectively. The second term

$$t_2 := ((\varphi \times \psi) \mid (P \times Q)_{\hat{1}})((\delta_{P\hat{1}} \times \eta_{Q\hat{1}}) \mid (P \times Q)_{\hat{1}})$$

satisfies

$$\begin{aligned} t_2((x_1, y_1), (x_2, y_2)) &= \sum_{\substack{(x, y) \in (P \times Q)_{\hat{1}} \\ x_1 \leqslant x \leqslant x_2 \\ y_1 \leqslant y \leqslant y_2}} (\varphi \times \psi)((x_1, y_1), (x, y))(\delta_{P\hat{1} \times \eta_{Q\hat{1}}}((x, y), (x_2, y_2))) \\ &= \sum_{\substack{(x_1, y) \in (P \times Q)_{\hat{1}} \\ y_1 \leqslant y < y_2}} \varphi(x_1, x_2)\, \psi(y_1, y) = \varphi(x_1, x_2) \sum_{\substack{(x_2, y) \in (P \times Q)_{\hat{1}} \\ y_1 \leqslant y < y_2}} \alpha_Q^{-1}(y_1, y). \end{aligned}$$

Consider the set $A(x_2, y_2) := \{ y \in Q_{\hat{1}} \mid y_1 \leqslant y < y_2,\ (x_2, y) \in (P \times Q)_{\hat{1}} \}$. Since $y < y_2$ implies $y \in Q$, there are two possibilities:

(i) $x_2 = \hat{1}$, then $A(x_2, y_2) = \varnothing$ and $t_2((x_1, y_1), (x_2, y_2)) = 0$.

(ii) $x_2 \neq \hat{1}$, then $A(x_2, y_2) = \{ y \in Q_{\hat{1}} \mid y_1 \leqslant y < y_2 \}$ and

$$t_2((x_1, y_1), (x_2, y_2)) = \varphi(x_1, x_2)\, r^Q(y_1, y_2).$$

Hence

$$t_2((x_1, y_1), (x_2, y_2)) = \eta_{P\hat{1}}(x_2, \hat{1})\, \varphi(x_1, x_2)\, r^Q(y_1, y_2). \tag{5.2.6}$$

Analogously we obtain for the third term

$$t_3 := ((\varphi \times \psi) \mid (P \times Q)_{\hat{1}})((\eta_{P\hat{1}} \times \delta_{Q\hat{1}}) \mid (P \times Q)_{\hat{1}})$$

$$t_3((x_1, y_1), (x_2, y_2)) = \eta_{Q\hat{1}}(y_2, \hat{1})\, r^P(x_1, x_2)\, \psi(y_1, y_2). \tag{5.2.7}$$

Since $(x_2, y_2) \in (P \times Q)_{\hat{1}}$ implies $(x_2 = \hat{1} \leftrightarrow y_2 = \hat{1})$ we have

$$\eta_{P\hat{1}}(x_2, \hat{1}) = \eta_{Q\hat{1}}(y_2, \hat{1}) = \eta_{(P \times Q)\hat{1}}(x_2, y_2), (\hat{1}, \hat{1})$$

and combining (5.2.4), (5.2.2), (5.2.5), (5.2.6), and (5.2.7) it is proved:

5.3. LEMMA. *Under the assumption of Proposition* 5.2 *it holds that for* $x_1, x_2 \in P_{\hat{1}}$, $y_1, y_2 \in Q_{\hat{1}}$,

$$\begin{aligned} r^{P \times Q}&((x_1, y_1), (x_2, y_2)) \\ &= r^P(x_1, x_2)\, r^Q(y_1, y_2) + \eta_{(P \times Q)\hat{1}}((x_2, y_2), (\hat{1}, \hat{1}))(\varphi(x_1, x_2)\, r^Q(y_1, y_2) \\ &\quad + r^P(x_1, x_2)\, \psi(y_1, y_2)). \end{aligned}$$

With $(x_2, y_2) = (\hat{1}, \hat{1})$ *we obtain again Proposition* 5.2.

As a consequence of 5.2 we get the following theorem:

5.4. THEOREM. *Let* $P=(P,\alpha_P,v_P)$ *and* $Q=(Q,\alpha_Q,v_Q)$ *be angular posets. For all j it holds that*

$$R_j(P\times Q)=\sum_i R_i(P)\,R_{j-i}(Q).$$

Proof. Since $\mathrm{rk}(x,y)=\mathrm{rk}(x)+\mathrm{rk}(y)$ we obtain from 5.2 that

$$\begin{aligned}
R_j(P\times Q) &= \sum_{\substack{(x,y)\\ \mathrm{rk}(x,y)=j}} v_{P\times Q}(x,y)\,r^{P\times Q}(x,y)\\
&= \sum_{\substack{(x,y)\\ \mathrm{rk}(x)+\mathrm{rk}(y)=j}} v_P(x)\,v_Q(y)\,r^P(x)\,r^Q(y)\\
&= \sum_{i=0}^{j}\Bigg(\sum_{\mathrm{rk}(x)=i} v_P(x)\,r^P(x)\Bigg)\Bigg(\sum_{\mathrm{rk}(y)=j-i} v_Q(y)\,r^Q(y)\Bigg)\\
&= \sum_{i=0}^{j} R_i(P)\,R_{j-i}(Q).
\end{aligned}$$

5.5. COROLLARY. *Under the assumptions of* 5.5 *it holds that for all j*

$$R^j(P\times Q)=\sum_i R^i(P)\,R^{j-i}(Q).$$

Proof.

$$\begin{aligned}
R^j(P\times Q) &= R_{\mathrm{rk}(P\times Q)-j}(P\times Q)=\sum_{i=0}^{\mathrm{rk}(P\times Q)-j} R_i(P)\,R_{\mathrm{rk}(P\times Q)-j-i}(Q)\\
&= \sum_{i=0}^{\mathrm{rk}(P)+\mathrm{rk}(Q)-j} R^{\mathrm{rk}(P)-i}(P)\,R^{\mathrm{rk}(Q)-(\mathrm{rk}(P)+\mathrm{rk}(Q)-j-i)}(Q)\\
&= \sum_{m=j-\mathrm{rk}(Q)}^{\mathrm{rk}(P)} R^m(P)\,R^{j-m}(Q)=\sum_i R^i(P)\,R^{j-i}(Q).
\end{aligned}$$

6. SUBDIVISIONS

6.1. Let $P=(P,\alpha_P,v_P)$ and $Q=(Q,\alpha_Q,v_Q)$ be angular posets. An isotone map $\varphi\colon P\to Q$ is called a subdivision, written $\varphi\colon P\lhd Q$ if it satisfies the following properties:

6.1.1. For all $x\in P$ there is an $z\in P$ with $x\leqslant z$, $\varphi(x)=\varphi(z)$, and $\mathrm{rk}(z)=\mathrm{rk}(\varphi(x))$.

6.1.2. $\sum_{x \in \varphi^{-1}(y),\, \mathrm{rk}(x)=\mathrm{rk}(y)} v_P(x) = v_Q(y)$ for all $y \in Q$.

6.1.3. If $x \in P$, $y \in Q$, and $\varphi(x) \leqslant y$ then

$$\sum_{\substack{z \in \varphi^{-1}(y) \\ x < z \\ \mathrm{rk}(z)=\mathrm{rk}(y)}} \alpha_P(x, z) = \alpha_Q(\varphi(x), y).$$

Remark that 6.1.3 can also be formulated in the form

6.1.3′. If $x \in P$, $y \in Q$, and $\varphi(x) \leqslant y$ then

$$\sum_{\substack{z \in \varphi^{-1}(y) \\ x < z \\ \mathrm{rk}(z)=\mathrm{rk}(y)}} \alpha_P(x, z) = \alpha_Q(\varphi(x), y).$$

This is because x, arising as an z in the sum of 6.1.3, implies $\varphi(x) = y$, $\mathrm{rk}(x) = \mathrm{rk}(y)$ and $x < z$ implies $\mathrm{rk}(z) > \mathrm{rk}(x) = \mathrm{rk}(y)$. Therefore in this case automatically

$$\sum_{\substack{z \in \varphi^{-1}(y) \\ x < z \\ \mathrm{rk}(z)=\mathrm{rk}(y)}} \alpha_P(x, z) = \alpha_P(x, x) = 1 = \alpha_Q(\varphi(x), y).$$

6.2. Example. Let L be a cell complex which is a subdivision of the cell complex K (see 1.11). In 1.11 we considered an isotone mapping φ from the poset $(L, \leqslant)$ into the poset $(K, \leqslant)$. Proposition 1.13 expresses exactly the fact that $\varphi\colon L \lhd K$ is a subdivision of the corresponding angular posets L and K (conditions 6.1.2 and 6.1.3; condition 6.1.1 is obviously satisfied). We will prove in the following that if: $P \lhd Q$ is a subdivision of angular posets the Lipschitz–Killing curvatures of P and Q coincide. In order to prove this we need the following lemma:

6.3. Lemma. *If $\varphi\colon P \lhd Q$ is a subdivision, then for all $x \in P$ it holds that: if $r^P(x) \neq 0$ then*

(i) $\mathrm{rk}(x) = \mathrm{rk}(\varphi(x))$ *and*

(ii) $r^P(x) = r^Q(\varphi(x))$.

The proof of this lemma will be prepared by some statements concerning a subdivision $\varphi\colon P \lhd Q$.

6.3.1. *φ is surjective; more precisely, to every $y \in Q$ there is an $x \in \varphi^{-1}(y)$ with* $\mathrm{rk}(x) = \mathrm{rk}(y)$.

Proof. Let $y \in Q$ be any element. By 6.1.2

$$\sum_{\substack{x \in \varphi^{-1}(y) \\ \mathrm{rk}(x) = \mathrm{rk}(y)}} v_P(x) = v_Q(y) \neq 0,$$

and this implies that there is at least an $x \in \varphi^{-1}(y)$ with $\mathrm{rk}(x) = \mathrm{rk}(y)$.

6.3.2. *Define now, for $x \in P$, $y \in Q$,*

$$\Theta(x, y) := \{z \in \varphi^{-1}(y) \cap P_{>y}, \mathrm{rk}(z) = \mathrm{rk}(y)\}$$

$$\theta(x, y) := \sum_{w \in \Theta(x, y)} \alpha_P(x, z).$$

Then

(a) $\theta(x, y) \neq 0 \Rightarrow \Theta(x, y) \neq \varnothing$.

(b) $\Theta(x, y) \neq \varnothing \Rightarrow \varphi(x) \leqslant y$ *and* $\theta(x, y) = \alpha_Q(\varphi(x), y)$.

(c) $\theta(x, \varphi(x)) \neq 0 \Leftrightarrow \Theta(x, \varphi(x)) \neq \varnothing \Leftrightarrow \theta(x, \varphi(x)) = 1$; $\varphi(x) < y \Rightarrow \alpha_Q(\varphi(x), y) = \theta(x, y)$.

Proof. (a) This is evident.

(b) If $z \in \Theta(x, y)$, then $\varphi(x) \leqslant z$, hence $\varphi(x) \leqslant \varphi(z) = y$. Moreover by 6.1.3

$$\alpha_Q(\varphi(x), y) = \sum_{\substack{z \in \varphi^{-1}(y) \\ x \leqslant z \\ \mathrm{rk}(z) = \mathrm{rk}(y)}} \alpha_Q(x, z) = \sum_{z \in \Theta(x, y)} \alpha_Q(x, z) + C$$

with

$$C = \begin{cases} \alpha_P(x, z) = 1 & \text{if } \varphi(x) = y \text{ and } \mathrm{rk}(x) = \mathrm{rk}(y) \\ 0 & \text{otherwise.} \end{cases}$$

Assume $C \neq 0$. Then $\varphi(x) = y$ and $\mathrm{rk}(x) = \mathrm{rk}(y)$. If $z \in \Theta(x, y)$, then $x < z$ whence $\mathrm{rk}(y) = \mathrm{rk}(x) < \mathrm{rk}(z)$, a contradiction!

(c) By (a) we have to prove only the direction $\Theta(x, y) \neq \varnothing \Rightarrow \theta(x, \varphi(x)) = 1$, which is clear by (b), because $\theta(x, \varphi(x)) = \alpha_Q(\varphi(x), \varphi(x)) = 1$.

(d) This follows directly from the proof of (b) because $\varphi(x) < y$ implies $C = 0$.

6.3.3. We have now all preliminaries to prove Lemma 6.3, which will be done by a top-down induction in the poset P. Assume $x \in P$ maximal.

By 6.3.2, $\varphi(x)$ is maximal in Q and $\mathrm{rk}(x)=\mathrm{rk}(\varphi(x))$, hence $r^P(x)=1=r^Q(\varphi(x))$. Assume now $x\in P$ and (i), (ii) being satisfied for all $z\in P$ with $x<z$. It will be proved that under these assumptions x satisfies (i) and (ii). In the following we are using the property * of Section 4 and (a), (b), (c), and (d) from 6.3.2:

$$\begin{aligned} r^P(x) &= 1-\sum_{x<z}\alpha_P(x,z)\,r^P(z)=1-\sum_{\substack{x<z\\ \mathrm{rk}(z)=\mathrm{rk}(\varphi(z))}}\alpha_P(x,z)\,r^Q(\varphi(z))\\ &=1-\sum_{y\in Q}\left(\sum_{z\in\Theta(x,y)}\alpha_P(x,z)\right)r^Q(y)=1-\sum_{\varphi(x)\leqslant y}\theta(x,y)\,r^Q(y)\\ &=1-\theta(x,\varphi(x))\,r^Q(\varphi(x))-\sum_{\varphi(x)<y}\alpha_Q(\varphi(x),y)\\ &=r^Q(\varphi(x))-\theta(x,\varphi(x))\,r^Q(\varphi(x))=r^Q(\varphi(x))(1-\theta(x,\varphi(x))). \end{aligned}$$

There are two possibilities:

Case 1. $\theta(x,\varphi(x))=0$. Then $r^P(x)=r^Q(\varphi(x))$ (ii) and by (c), $\Theta(x,\varphi(x))=\varnothing$. This implies $\mathrm{rk}(x)=\mathrm{rk}(\varphi(x))$ (i), for if $\mathrm{rk}(x)\neq\mathrm{rk}(\varphi(x))$ by 6.1.1 there is a $z\in P\mathrm{d}$ with $x\leqslant z$, $\varphi(x)=\varphi(z)$, and $\mathrm{rk}(z)=\mathrm{rk}(\varphi(x))$. Since $\mathrm{rk}(x)\neq\mathrm{rk}(\varphi(x))$, $z\neq x$; hence $x<z$ and therefore $z\in\Theta(x,\varphi(x))$, a contradiction.

Case 2. $\theta(x,\varphi(x))\neq 0$. By (c), $\theta(x,\varphi(x))=1$ and therefore $r^P(x)=0$ and $\Theta(x,\varphi(x))\neq\varnothing$. Assume $z\in\Theta(x,\varphi(x))$, then $x<z$ and $\mathrm{rk}(x)<\mathrm{rk}(z)=\mathrm{rk}(\varphi(x))$. Therefore if $r^P(x)\neq 0$ only Case 1 can arise in which (i) and (ii) are satisfied.

This concludes the proof of Lemma 6.3. From Lemma 6.3 we get directly the following theorem:

6.4. THEOREM. *If $\varphi\colon P\lhd Q$ is a subdivision then for all j it holds that*

$$R_j(P)=R_j(Q)$$

and

$$R^j(P)=R^j(Q).$$

Proof. The second statement follows from the first because $\mathrm{rk}(P)=\mathrm{rk}(Q)$ by 6.3.2. By 6.1.2, 6.3.1, and 6.3 we get

$$\begin{aligned} R_j(P) &= \sum_{\mathrm{rk}(x)=j}\nu_P(x)\,r^P(x)=\sum_{\mathrm{rk}(x)=j=\mathrm{rk}(\varphi(x))}\nu_P(x)\,r^Q(\varphi(x))\\ &=\sum_{\substack{y\in Q\\ \mathrm{rk}(y)=j}}\left(\sum_{\substack{\mathrm{rk}(x)=j\\ \varphi(x)=y}}\nu_P(x)\right)r^Q(y)=\sum_{\substack{y\in Q\\ \mathrm{rk}(y)=j}}\nu_Q(y)\,r^Q(y)=R_j(Q). \end{aligned}$$

7. Lipschitz–Killing Curvatures of Polyhedrons

7.1. Assume we are given two functions α and v, where α assigns to every pair C, D of cells a real number $\alpha(C, D)$ and v assigns to every cell C a real number $v(C)$. Assume furthermore that these functions are subject to the following conditions:

(i) If $\alpha(C, D) \neq 0$ then C is a face of D.

(ii) $\alpha(C, C) = 1$ and $v(C) \neq 0$ for al cells C.

(iii) Let P be a cell and let L be a cell complex which is a subdivision of $\mathscr{F}(P)$. Let $\varphi: L \to \mathscr{F}(P)$ be the corresponding mapping (see 1.10).

(1) Then

$$\sum_{\substack{C \in L \\ \dim C = n}} \operatorname{vol}(C) = \operatorname{vol}(P).$$

(2) If $B \in L$ then

$$\sum_{\substack{B \subseteq C \in L \\ \dim C = n}} \alpha(B, C) = \alpha(\varphi(B), P).$$

(iv) If A, B, C, D are cells, then $\alpha(C \times D, A \times B) = \alpha(C, D)\, \alpha(D, B)$.

7.2. If K is a cell complex, then $K = (K, \alpha \downarrow K^2, v \downarrow K)$ is an angular poset. Hence the Lipschitz–Killing curvatures $R_j(K)$ are defined. If K and L are cell complexes with $|L| = |K|$ then there is a cell complex M with $M \lhd L$, $M \lhd K$. Hence by 6.4, $R_j(L) = R_j(M) = R_j(K)$ and this allows us to define:

Let P be a polyhedron and let L be a cell complex with $|L| = P$ then

$$R_j(P) := R_j(L).$$

Because $|L \times K| = |L| \times |K|$ we get from 5.3 that

$$R_j(P \times Q) = \sum_i R_i(P)\, R_{j-i}(Q)$$

for all polyhedrons P and Q.

If α is the interior angle (see 1.6) and if v is the volume, then $R_j(P)$ is the ordinary Lipschitz–Killing curvature of the polyhedron P.

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